CATEGORY OF FUZZY HYPER BCK-ALGEBRAS

J.DONGHO*

Abstract. In this paper we first define the category of fuzzy hyper BCK-algebras. After that we show that the category of hyper BCK-algebras has equalizers, coequalizers, products. It is a consequence that this category is complete and hence has pullbacks.

1. Introduction

The study of hyperstructure was initiated in 1934 by F. Marty at 8th congress of Scandinavian Mathematiciens. Y.B. Jun et al. applied the hyperstructures to BCK-algebras, and introduces the notion of hyper BCK-algebra. Now we follow [1,2,3,4] and introduce the category of fuzzy hyperBCK-algebra and obtain some result, as mentioned in the abstarct.

2. Preliminaries

We now review some basic definitions that are very useful in the paper.

Definition 1. [3] *Let H be an non empty set.*
A hyperoperation $*$ on H is a mapping of $H \times H$ family of non-empty subsets of H $\mathcal{P}^*(H)$

Definition 2. *Let $*$ be an hyperoperation on H and O a constant element of H An hyperorder on H is subset $<$ of $\mathcal{P}^*(H) \times \mathcal{P}^*(H)$ define by:*
for all $x, y \in H, x < y$ iff $O \in x * y$ and for every $A, B \subseteq H, A < B$ iff $\forall a \in A, \exists b \in B$ such that $a < b$.

Definition 3. *If $*$ is hyperoperation on H.*
For all $A, B \subseteq H, A * B := \bigcup_{a \in A, b \in B} a * b$

Definition 4. [1] *By hyper BCK-algebra we mean a non empty set H endowed with a hyper-operation $*$ and a constant O satisfying the following axioms.*

(HK1) $(x * z) * (y * z) < (x * y)$
(HK2) $(x * y) * z = (x * z) * y$
(HK3) $x * H < \{x\}$

Definition 5. *A fuzzy hyper BCK-algebra is a pair $(\mathbf{H}; \mu_H)$ where $\mathbf{H} = (H; *; O)$ is hyper BCK-algebra and $\mu_H : H \longrightarrow [0, 1]$ is a map satisfy the following property:*

$$\inf(\mu_H(x * y)) \geq \min(\mu_H(x), \mu_H(y))$$

for all $x, y \in H$.

Example 1. [5] *Let $n \in \mathbb{N}^*$. Define the hyperoperation $*$ on $H = [n, +\infty)$ as follows:*

$$x * y = \begin{cases} [n, x] & \textit{iff} \quad x < y \\ (n, y] & \textit{iff} \quad x > y \neq n \\ \{x\} & \textit{iff} \quad y = n \end{cases}$$

for all $x, y \in H$. To show that $(H, *, n)$ is hyper BCK-algebra, it suffice to show axiom $HK3$. For all $x \in H, x * H = \bigcup_{t \in H} x * t$. For all $x \in H$ then $x * x \subseteq x * H$. And then $n \in [n, x] * \{x\}$

3. The category of fuzzyhyper BCK-algebras

Lemma 1. *Let $(\mathbf{H}; \mu_H)$ be a fuzzy hyper BCK-algebra.*
For all $x \in H, \mu_H(O) \geq \mu_H(x)$

Proof. For all $x \in H, x < x$; then $O \in x * x$.

$$\begin{array}{lll} O \in x * x & \texttt{imply} & \mu_H(O) \geq \inf(\mu_H(x * y)) \geq \min(\mu_H(x), \mu_H(y)) \\ & \texttt{i.e} & \mu_H(O) \geq \min(\mu_H(x), \mu_H(y)) = \mu_H(x) \\ & \texttt{i.e} & \mu_H(O) \geq \mu_H(x). \end{array}$$

■

Definition 6. *Let $(\mathbf{H}; \mu_H)$ be a fuzzy hyperBCK-algebra. μ_H is called a fuzzy map.*

Lemma 2. *Let $(\mathbf{H}; \mu_H)$ a fuzzy hyper BCK-algebra.The following properties are trues:*

i) *If for all $x, y \in H, x < y$ imply $\mu_H(x) \leq \mu_H(y)$ then for all $x \in H, \mu_H(x) = \mu_H(O)$*
ii) *If $\mu_H(O) = 0$ then $\mu_H(x) = 0$*

Proof.

i) For all $x \in H$, $x * H < \{x\}$ then $x * O < x$.
$O < x \Rightarrow \mu_H(O) \leq \mu_H(x)$.
Then $\mu_H(x) \leq \mu_H(O)$ and $\mu_H(O) \leq \mu_H(x)$ for all $x \in H$.
i.e $\mu_H(x) = \mu_H(O)$ for all $x \in H$.
ii) $\mu_H(O) = O \Rightarrow \mu_H(O) \leq \mu_H(x)$, for all $x \in H$.
Then $\mu_H(x) = \mu_H(O)$ for all $x \in H$.

■

Definition 7. *Let $(\mathbf{H}; \mu_H)$ and $(\mathbf{F}, \mu_F)$ two fuzzy hyperBCK-algebras. An homomorphism from $(\mathbf{H}, \mu_H)$ to $(\mathbf{F}, \mu_F)$ is an homomorphism $f : \mathbf{H} \longrightarrow \mathbf{F}$ of hyper BCK-algebra such that for all $x \in H$, $\mu_F(f(x)) \geq \mu_H(x)$*

Proposition 1. *Let $(\mathbf{H}, \mu_H)$ an hyperBCK-algebra. Let $\mathbf{G}, \mathbf{F} \subset H$ two hyperBCK-sualgebras of $\mathbf{H}$. If there exist $\alpha \in]0, 1[$ such that $\mu_H(G^*) \subset [0, \alpha[$ and $\mu_H(F) \subseteq]\alpha, 1]$. Then any homorphism of hyper BCK-algebra $f : G \longrightarrow F$ is homomorphism of fuzzy hyperBCK-algebra.*

Proof. Suppose that there is $\alpha \in]0,1]$ such that
$\mu_H(G^*) \subset [0,\alpha[$ and $\mu_H(F) \subseteq]\alpha, 1[$.
Let $f : G \longrightarrow F$ an homomorphism of hyper BCK-algebra.
For all $x \in G^*, f(x) \in F$. And $\mu_F(f(x)) > \alpha > \mu_H(x)$.
Then $\mu_F(f(x)) > \mu_H(x)$ for all $x \in G^*$
$f(O) = O$ then $\mu_F(f(O)) = \mu_F(O) = \mu_H(O)$
i.e $\mu_F(f(O)) = \mu_H(O)$. therefore, for all $x \in x \in G, \mu_F(x) \geq \mu_H(x)$ ∎

Example 2. [1] *Define the hyper operation* " $*$ " *on* $H = [1; +\infty]$ *as follow.*

$$x * y = \begin{cases} [1,x] & \textbf{\textit{if}} \quad x \leq y \\ (1,y] & \textbf{\textit{if}} \quad x > y \neq 1 \\ \{x\} & \textbf{\textit{if}} \quad y = 1 \end{cases}$$

For all $x, y \in H, (\mathbf{H}, *, 1)$ *is hyperBCK-algebra. Define the fuzzy structure* μ_H *on* H *by:*

$$\begin{array}{rccc} \mu_H : & H & \longrightarrow & [0,1] \\ & x & \mapsto & \frac{1}{x} \end{array}$$

We show that $(\mathbf{H}, \mu_H)$ *is a fuzzy hyper BCK-algebra.*
Let $x, y \in H$.

(i) *If* $x \leq y$, *then* $x * y = [1, x]$; *i.e for all* $t \in x * y, 1 \leq t \leq x \leq y$ *and so* $\frac{1}{y} \leq \frac{1}{x} \leq \frac{1}{t}$. *So,* $\mu_H(t) \geq \frac{1}{y} = \min\{\frac{1}{y}, \frac{1}{x}\} = \min\{\mu_H(x), \mu_H(y)\}$. *Then* $\inf\{x * y\} \geq \min\{\mu_H(x), \mu_H(y)\}$

(ii) *If* $x > y \neq 1$ *then* $x * y = (1, y]$. *For all* $t \in H \cap x * y, \frac{1}{x} \leq \frac{1}{y} \leq \frac{1}{t} \leq 1$. *therefore,* $\mu_H(t) = \frac{1}{t} \geq \frac{1}{x} = \min\{\mu_H(x), \mu_H(y)\}$ *for all* $t \in x * y$. *Then* $\in \{\mu_H(x * y)\} \geq \min\{\mu_H(x), \mu_H(y)\}$.

(iii) *If* $y = 1, x * y = \{x\}$, *hence* $\mu_H(x * y) = \{\mu_H(x)\} = \{\frac{1}{x}\}$.
$y = 1$*imply* $y \leq x$ *and* $\frac{1}{x} \leq \frac{1}{y}$ *for all* $x \in H$*; i.e;*
$\min\{\mu_H(x), \mu_H(y)\} = \frac{1}{x}$. *Then* $\mu_H(x * y) = \{\frac{1}{x}\}$.
Thus $\inf\{\mu_H(x * y)\} = \frac{1}{x} \geq \min\{(\mu_H(x), \mu_H(y))\}$

Proposition 2. *The fuzzy hyperBCK-algebras and homomorphisms of fuzzy hyperBCK-algebras form a category.*

Proof. The proof is easy. ∎

Notes 1. *In the following we let* $\mathcal{H}$ *the category of hyperBCK-algebras;* $\mathbb{F}_{\mathcal{H}}$ *the category of fuzzy hyperBCK-algebras;* $\mathbb{H}$ *the fuzzy hyper BCK-algebra* $(\mathbf{H}, \mu_H)$

For any fuzzy hyper BCK-algebra $\mathbb{H}$, we associate for all $\alpha \in [0,1]$ the set $H_\alpha := \{x \in H, \mu_H(x) \geq \alpha\}$

Lemma 3. *Let* $\mathbb{H}$ *a fuzzy hyper BCK-algebra. For all* $\alpha \in [0,1], O \in H_\alpha$ *and for all* $x, y \in H, x * y \subseteq H_\alpha$

Proof. By lemma 1, for all $x \in H, \mu_H(x) \leq \mu_H(0)$.
Then for all $x \in H_\alpha, \mu_H(O) \geq \mu_H(x) > \alpha$ i.e $O \in H_\alpha$.
Let $x, y \in H_\alpha$;

for all $t \in x * y$, $\mu_H(t) \geq \inf\{\mu_H(x * y) \geq \min\{\mu_H(x), \mu_H(y)\}\} \geq \alpha$
then $t \in H_\alpha$. therefore, $x * y \subseteq H_\alpha$
■

Definition 8. *Let $(H, *, O)$ be an hyper BCK-algebra. An hyper BCK-subalgebra of H is a non empty subset S of H such that $O \in S$ and S is hyper BCK-algebra with respect to the hyper operation " $*$ " on H*

Proposition 3. *Let $(H, *, O)$ be an hyper BCK-algebra. A non empty subset S of H is hyper BCK-subalgebra of H iff for all $x, y \in S, x * y \in S$*

Proof. The proof is easy. ■

Definition 9. *A fuzzy hyper BCK-subalgebra of $\mathbb{H}$ is an hyper BCK-subalgebra S of $\mathbf{H}$ with the restriction μ_S of μ_H on S.*

Proposition 4. *For all $\alpha \in [0, 1]$, (H_α, μ_H) is fuzzy hyper BCK-subalgebra of $\mathbb{H}$*

Proof. By lemma 3, H_α is hyper BCK-subalgebra of $\mathbf{H}$
and $\inf\{\mu_H(x * y)\} \geq \min\{\mu_H(x), \mu_H(y)\}$ ■

Definition 10. *Let $\mathbb{H}$ by an fuzzy hyper BCK-algebra. The fuzzy-hyperBCK-subalgebra $\mathbf{H}_\alpha := (H_\alpha; \mu_H)$ is calling hyper α-cut of $\mathbb{H}$*

Proposition 5. *Let $\mathbb{H}$ be fuzzy hyper BCK-algebra. A hyper BCK-subalgebra S of $\mathbf{H}$ is fuzzy hyper BCK-subalgebra iff S is hyper α-cut of H.*

Proof. By prosition 4, any hyper α-cut is fuzzy hyper BCK-subalgebra.
Conversely, let S be fuzzy hyper BCK-subalgebra of $\mathbb{H}$. Then $\mu_H(S)$ is subset of $[0, 1]$.
If $0 \in \mu_H(S)$, then $S = H_0 = \mathbb{H}$.
If $0 < \inf(\mu_H(S))$, then $S = H_{\inf(\mu_H(S))}$. ■

Proposition 6. *Let $\mathbb{H}$ and $\mathbb{F}$ be two fuzzy hyper BCK algebras. An $\mathcal{H}$-morphism $f : H \longrightarrow F$ is $\mathbb{F}_{\mathcal{H}}$-morphism iff for all $\alpha \in [0, 1], f(H_\alpha) \subseteq F_\alpha$.*

Proof. Suppose that $f(H_\alpha) \subseteq F_\alpha$ for all $\alpha \in [0, 1]$ Let $x \in [0, 1]$ we need $\mu_H(x) \leq \mu_F(f(x))$. Let $\alpha = \mu_H(x)$; $\quad x \in H_\alpha$ and $f(x) \in f(H_\alpha) \subseteq F_\alpha$. Then $\mu_F(f(x)) > \alpha = \mu_H(x)$. whence for all $x \in H, \mu_F(f(x)) \geq \mu_H(x)$.
Conversely, suppose that $f : \mathbb{H} \longrightarrow \mathbb{F}$ is $\mathbb{F}_{\mathcal{H}}$-morphism.
For all $x \in H_\alpha$ for some $\alpha \in [0, 1]$, $\mu_F(f(x)) \geq \mu_H(x) \geq \alpha$i.e; $f(x) \in [0, 1]$.
Then $f(H_\alpha) \subseteq F_\alpha$ for all $\alpha \in [0, 1]$.
■

Proposition 7. *A $\mathbb{F}_{\mathcal{H}}$-morphism $f : \mathbb{H} \longrightarrow \mathbb{F}$ is $\mathbb{F}_{\mathcal{H}}$-iso iff it is both $\mathcal{H}$-iso and $\mu_H = \mu_F f$.*

Proof. Suppose that f is $\mathcal{H}$-iso and $\mu_H = \mu_F f$. there is $g \in Hom_{\mathcal{H}}(\mathbf{F}, \mathbf{H})$; $g \circ f = Id_H$ and $g \circ g = Id_F$.
Then, for all $x \in F, \mu_H(g(x)) = \mu_F(f(g(x)))\mu_F(x)$.
And then, $g \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{F}, \mathbb{H})$.

Conversely, Suppose that f is $\mathbb{F}_{\mathcal{H}}$-iso.
There is $g \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{F}, \mathbb{H}); g \circ f = Id_F$ and $f \circ g = Id_H$.
Since $f \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{F}, \mathbb{H}), \mu_H \leq \mu_F f$.
Since $g \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{H}, \mathbb{H}), \mu_F \leq \mu_H g$. $x \in H$ imply $f(x) \in F$. Then $\mu_F(f(x)) \leq \mu_H(g(f(x))) = \mu_H(x)$ i.e; $\mu_F f \leq \mu_H$.
therefore, $\mu_F f = \mu_H$ ■

Proposition 8. *Let* $f \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{F}, \mathbb{H})$.
f *is* $\mathbb{H}_{\mathcal{H}}$*-mono iff* f *is* $\mathcal{H}$*-mono*

Proof. Suppose that f is $\mathbb{H}_{\mathcal{H}}$-mono.
For all $h, g \in Hom_{\mathcal{H}}(\mathbf{K}, \mathbf{H})$, such that $fh = fg$, we define $\mu_K = \min(\mu_H(h(x); \mu_H(g(x))))$ for all $x \in K$.

a) We show that $(K; \mu_K)$ is fuzzy hyper BCK-algebra.

$$\begin{aligned}
\inf(\mu_K(x * y)) &= \inf\{\mu_H(h(x * y); \mu_H(g(x * y)))\} \\
&= \inf\{\mu_H(h(x) * h(y); \mu_H(g(x) * g(y)))\} \\
&= \min\{\inf\{\mu_H(h(x) * h(y)\}; \inf\{\mu_H(g(x) * g(y)))\}\} \\
&\geq \min\{\min\{\mu_H(h(x); \mu_H(h(y)\}; \min\{\mu_H(g(x); g(y))\}\} \\
&\geq \min\{\min\{\mu_K(x); \mu_K(y)\}\} \\
&\geq \min\{\mu_K(x); \mu_K(y)\}
\end{aligned}$$

Then, for all $x, y \in K, \inf(\mu_K(x * y)) \geq \min\{\mu_K(x), \mu_K(y)\}$. therefore, $(K; \mu_K)$ is fuzzy hyper BCK-algebra.

b) We show that h and g are $\mathbb{F}_{\mathcal{H}}$-homomorphism.
For all $x \in K$, $\mu_K(x) = \min\{\mu_H(h(x), \mu_H(g(x)))\}$.
Then $\mu_K(x) \leq \mu_H(g(x))$ and $\mu_K(x) \leq \mu_H(h(x))$.
therefore, h and g are $\mathbb{F}H$-morphism.
Since f is $\mathbb{F}_{\mathcal{H}}$-mono and $h, g \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{F}, \mathbb{H})$, $fh = fg$ imply $f = g$
Conversely, if f is $\mathbb{F}_{\mathcal{H}}$-mono, it is $\mathcal{H}$-mono.

■

Lemma 4. *The pair* $\mathbb{O} = (\{O\}, \mu_o)$ *where*

$$\begin{array}{cccc}
\mu_o: & \{o\} & \longrightarrow & [0,1] \\
 & o & \mapsto & 0
\end{array}$$

is fuzzy hyper BCK-algebra

Proof. Easy ■

Lemma 5. $\mathbb{O}$ *is final objet of* $\mathbb{F}_{\mathcal{H}}$

Proposition 9. *The category* $\mathbb{F}_{\mathcal{H}}$ *has products.*

Proof. Let $(\mathbb{H}_i; \mu_{H_i})_{i \in I}$ a family of fuzzy hyper BCK-algebras.
Denote $\mathbf{H} = \prod_{i \in I} H_i$ the $\mathcal{H}$-product of $(H_i)_{i \in I}$ with the projection morphisms $p_i : \mathbf{H} \longrightarrow H_i$. Consider the following map $\mu_H : H \longrightarrow [0, 1]$ define by:

$$\mu_H(x) = \bigwedge_{i \in I} \mu_{H_i} p_i(x)$$

for all $x \in H$

a) We show that the pair $(\mathbf{H}; \mu_H)$ is fuzzy hyper BCK-algebra.
For all $x, y \in H, p_i(x * y) = p_i(x) * p_i(y)$ for all $i \mathbb{N} I$.
Then

$$\begin{array}{rcl} \inf(\mu_{H_i} p_i(x * y)) & = & \inf(\mu_{H_i}(p_i(x) * p_i(y)) \\ & \geq & \min\{\mu_{H_i}(p_i(x)); \mu_{H_i}(p_i(y))\} \end{array}$$

for all $i \in I$.
Then,

$$\begin{array}{rcl} \inf(\bigwedge_{i \in I} \mu_{H_i} p_i(x * y)) & \geq & \bigwedge_{i \in I} \inf\{\mu_{H_i}(p_i(x) * p_i(y)\} \\ & \geq & \bigwedge_{i \in I} \min\{\mu_{H_i}(p_i(x)); \mu_{H_i}(p_i(y))\} \\ & \geq & \min\{\bigwedge_{i \in I} \mu_{H_i} p_i(x); \mu_{H_i} p_i(y)\} \\ & \geq & \min\{\mu_H(x), \mu_H(y)\} \end{array}.$$

b) For all $i \in I, x \in H; \mu_{H_i} p_i(x) \geq (\bigwedge_{i \in I} \mu_{H_i} p_i)(x)$.
Then each p_i is $\mathbb{F}_{\mathcal{H}}$-morphism.

c) If $q_j : \mathbb{F} \longrightarrow \mathbb{H}_j$ is family of $\mathbb{F}_{\mathcal{H}}$-morphism, there is unique $\mathcal{H}$-morphism $\varphi : \mathbf{F} \longrightarrow \mathbf{H}$ such that the following diagram commute.

$$\begin{tikzcd} H \arrow[r, "p_j"] & H_j \\ F \arrow[u, "\varphi"] \arrow[ur, "q_j"'] & \end{tikzcd}$$

i.e $p_j \varphi = q_j$ for all $j \in I$
For all $x \in F$, $\mu_F(x) \leq \mu_{H_i} q_i(x)$.

$$\begin{array}{lll} \texttt{Then} & \mu_F(x) & \leq \mu_{H_i} p_i \varphi(x) \quad \texttt{for all} \quad x \in F, i \in I \\ \texttt{i.e} & \mu_F(x) & \leq \bigwedge_{i \in I} \mu_{H_i} p_i \varphi(x) \\ & & \leq (\bigwedge_{i \in I} \mu_{H_i} p_i) \varphi(x) \\ & & \leq \mu_H(\varphi(x)) \quad \texttt{for all} \quad x \in F. \\ & \texttt{Then} & \mu_F \leq \mu_H \varphi \leq \end{array}$$

Then φ is $\mathbb{F}_{\mathcal{H}}$-morphism.

■

Proposition 10. $\mathbb{F}_{\mathcal{H}}$ *have equalizers.*

Proof. Let $f, g \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{H}, \mathbb{F}), K := \{x \in H, f(x) = g(x)\}$.
It is prove in [1] that K is hyper BCK-subalgebra of H. It is clear that (K, μ_H) is fuzzy hyper BCK-algebra. Let $i : K \longrightarrow H$ the inclusion map. $i \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{K}, \mathbb{F})$. For all $x \in K, fi(x) = f(x) = g(x) = gi(x)$.
Let $h \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{L}, \mathbb{F})$ such that $fh = gh$, for all $x \in L, f(h(x)) = g(h(x))$.

Then $Imh \subseteq L$. Define $\delta : L \longrightarrow K$ by $\delta(x) = h(x)$ for all $x \in L$. $\delta \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{H}, \mathbb{F})$ and $i\delta = h$. So, the following diagram commute.

$$\begin{tikzcd} \mathbb{K} \arrow[r, "i"] & \mathbb{H} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{F} \\ \mathbb{L} \arrow[u, "\delta"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Since i is monic, δ is unique $\mathbb{F}_{\mathcal{H}}$-morphism such that the above diagram commute.
therefore, $\mathbb{F}_{\mathcal{H}}$ have equalizers. ■

Proposition 11. $\mathbb{F}_{\mathcal{H}}$ *is complet.*

Proof. By proposition 9, each family of objets of $\mathbb{F}_{\mathcal{H}}$ has product.
By proposition 10, each pair of parallel arrows has an equalizer. Then $\mathbb{F}_{\mathcal{H}}$ is complet. ■

Corollary 1. $\mathbb{F}_{\mathcal{H}}$ *has pulbacks*

Proof. By propositions 9 and 10, $\mathbb{F}_{\mathcal{H}}$ has equalizers and products. therefore, $\mathbb{F}_{\mathcal{H}}$ has pulbacks. ■

Proposition 12. $\mathbb{F}_{\mathcal{H}}$ *have coequalizers*

Proof. Let $f, g \in Hom_{\mathbb{F}_{\mathcal{H}}}(\mathbb{H}, \mathbb{K})$. Let

$$\sum_{fg} = \{\theta, \theta \ \ \texttt{regular congruence relation on} \ \ \mathbf{K} \ \ \texttt{such that} \ \ f(a)\theta g(a) \forall a \in H\}$$

$\sum_{fg} \neq \phi$ because $K \times K \in \sum_{fg}$
Let $\rho = \bigcap_{\theta \in \sum_{fg}} \theta$. Then, ρ is regular congruence relation.
Define on K/ρ the following hyper operation

$$[x]_\rho * [y]_\rho = [x * y]_\rho.$$

$(K/\rho; *; [0]_\rho)$ is an objet of $\mathcal{H}$ (see [1]).
Define on K/ρ the following map

$$\begin{array}{llll} \mu_{K/\rho}: & K/\rho & \longrightarrow & [0,1] \\ & \mu_{K/\rho}([x]_\rho) & \longmapsto & \bigvee_{a \in [x]_\rho} \mu_K(a) \end{array}$$

a) We show that $(K/\rho, \mu_{K/\rho})$ is objet of $\mathbb{F}_{\mathcal{H}}$.
If $x, y \in K$ such that $[x]_\rho = [y]_\rho$.
Then

$$\bigvee_{a \in [x]_\rho} \mu_K(a) = \bigvee_{a \in [y]_\theta} \mu_K(a)$$

$\forall x \in K$, $\mu_K(x) \leq \bigvee_{a \in [x]_\rho} \mu_K(a)$.
Then

$$\mu_K \leq \mu_{K/\rho}([x]_\rho) = \mu_{K/\rho}(\pi(x))$$

Then, the canonical projection π is an $\mathbb{F}_{\mathcal{H}} - morphism$
Since for all $x \in H, f(x)\rho g(x)$, then $[f(x)]_\rho = [g(x)]_\rho$.
therefore, $(\pi \circ f)(x) = (\pi \circ g)(x)$.
Then, $\pi \circ f = \pi \circ g$.

b) Universal property of coequalizer.
Let $\varphi : \mathbb{K} \longrightarrow \mathbb{L}$ and $\mathbb{F}_{\mathcal{H}}$-morphism such that $\varphi \circ f = \varphi \circ g$.
Define the following mapping.

$$\begin{array}{rccc} \psi : & K/\rho & \longrightarrow & L \\ & [x]_\rho & \longmapsto & \varphi(x) \end{array}$$

c) We prove that ψ is well define.
If $[x]_\rho = [y]_\rho$ then, for all $a \in H, \varphi(f(a)) = \varphi(g(a))$ imply $f(a)R_\varphi g(a)$ because R_φ is regular congruence on K. Then $R_\varphi \in \sum_{f,g}$. The minimality of ρ on $\sum_{f,g}$ imply $\rho \subseteq R_\varphi$.
therefore, $[x]_\rho = [y]_\rho$ imply $x\rho y$.
Then $xR_\varphi y$. i.e $\varphi(x) = \varphi(y)$
And then, $\psi([x]_\rho) = \psi([y]_\rho)$ therefore, ψ is well define.
For all $x \in K, \mu_L(\psi(\pi)(x)) = \mu_L(\varphi(x)) \geq \mu_K(x), \forall x \in K$ then for all $a \in [x]_\rho$. $\mu_L(\varphi(a)) \geq \mu_K(a)$. By the minimality of ρ, $[a]_\rho = [x]_\rho$ imply $a\rho x$ then $aR_\varphi x$ i.e $\varphi(a) = \varphi(x)$ (because $\rho \subseteq R_\varphi$). Then

$$\bigvee_{a \in [x]_\rho} \tilde{L}(\varphi(a)) = \bigvee_{a \in [x]_\rho} \tilde{L}(\varphi(x)) = \tilde{L}(\varphi(x))$$

therefore

$$\tilde{L}(\varphi(x)) \geq \bigvee_{a \in [x]_\rho} (a) = \tilde{K}/\rho([x]_\rho)$$

i.e $\tilde{L}(\psi([x]_\rho)) \geq \tilde{k}/\rho([x]_\rho)\forall x \in H$. It is clean that

$$\psi(\pi(x)) = \psi([x]_\rho) = \varphi(x), \forall x \in H$$

i.e $\psi \circ \pi = \varphi$
This prove the commutativity of the following diagram:

$$\begin{tikzcd} H \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & K \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & K/\rho \arrow[d, "\psi"] \\ & & L \end{tikzcd}$$

The unicity of ψ is thus to the fat that π is epimorphism.
Then, $\mathcal{F}_{\mathcal{H}}$ have coequalizer.

■

ACKNOWLEDGEMENTS

References

[1] H. Harizavi,J. Macdonald AND A. Borzooei *Category of bck-algebras*:Scientiae Mathematicae Japonicae Online, e-2006, 529-537.

[2] R. A. Borzooei H.Harizavi. *Regular Congruence Relations on hyper BCK-algebras*: vol.61, No.1(2005),83-97

[3] Eun Hwan ROH, B. Davvaz And Kyung Ho Kim *T-fuzzy subhypernea-rings of hypernear-ring* , Scientiae Mathematicae Japonicae Online, e-2005,19-29 .

[4] Carol L. Walker, *Category of fuzzy sets.*

[5] R.A. Borzoei and M.M. Zahedi, *Positive implicative hyperK-ideals*, Scientiae Mathematicae Japonicae Online, Vol. 4,(2001), 381-389.

[6] C. LELE and M. Salissou, *Discussiones Mathematicae, general Algebra and Application* **26**: 111-135(2006).

****Department of Mathematics, University of Yaounde, BP 812, Cameroon
E-mail address: josephdongho@yahoo.fr